EPFL - Autumn semester 2025-2026	J. Scherer
Cohomology rings	**Sheet 3**
Mathematics	26 Septembre 2025

This Monday is a holiday and so, this week's exercises sheet is shorter and only contains three possible assignments to conclude the first round of written exercises to hand in. They do not require new material.
Updated version following the discussion on Ed-forum. Thank you Qian for pointing out the mistake and Eliot and Kasimir for the discussion.

◊ **Exercice 1. A function chain complex.** Just like categories are enriched over themselves (there is a category of functors and natural transformations), and just like spaces are enriched over themselves, so are chain complexes. We define in this exercise a chain complex of chain maps. Let $C_\bullet, D_\bullet$ be two chain complexes of R-modules.
Define $\mathscr{H}om_R(C_\bullet, D_\bullet)_n$ to be the product $\prod_q \mathrm{Hom}_R(C_q, D_{q+n})$ and the differential is given by $d_n(f) = d^D_{n+q} \circ f + (-1)^n f \circ d^C_q$.
Recall that the tensor product $C_\bullet \otimes D_\bullet$ is given in degree n by $\bigoplus_{i+j=n} C_i \otimes D_j$ and the differential is given on the component $C_i \otimes D_j$ by $d^C_i \otimes id + (-1)^i id \otimes d^D_j$.

1. Verify that this defines a chain complex $\mathscr{H}om_R(C_\bullet, D_\bullet)$.
2. Show that $H_0(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong [C_\bullet, D_\bullet]$, the abelian group of homotopy classes of chain maps.
3. Show that $H_n(\mathscr{H}om_R(C_\bullet, D_\bullet)) \cong [C_\bullet[n], D_\bullet]$, the abelian group of homotopy classes of chain maps of degree n.
4. Verify that the pairing $\langle -, - \rangle \colon \mathscr{H}om_R(C_\bullet, D_\bullet) \otimes_R C_\bullet \to D_\bullet$ defined by $\langle f, c \rangle = f(c)$ is a chain map, i.e. $d^D \langle f, c \rangle = \langle df, c \rangle + (-1)^n \langle f, d^C c \rangle$.
5. Prove that composition $\mathscr{H}om_R(D_\bullet, E_\bullet) \otimes \mathscr{H}om_R(C_\bullet, D_\bullet) \to \mathscr{H}om_R(C_\bullet, E_\bullet)$ is a chain map (use the notation from point 4.).

◊ **Exercice 2. The normalized bar resolution.** The purpose of this exercise is to reduce the bar resolution by killing an "obviously" acyclic subcomplex in the standard bar resolution. Let $F_\bullet$ be the bar resolution, so $F_n = \mathbb{Z}[G^{n+1}]$. Let $D_n \subset F_n$ be generated by the elements $(g_0, \ldots, g_n)$ for which there exists an index i with $g_i = g_{i+1}$.

1. Verify that this defines a subcomplex $D_\bullet$ of $\mathbb{Z}G$-modules. Show that it is generated, as a free $\mathbb{Z}G$-module by the elements $[g_1 | \ldots | g_n]$ for which there exists an index i with $g_i = 1$.
2. Show that the contracting homotopy from Sheet 1, Exercise 3, restricts to $D_\bullet$, and therefore induces a contracting homotopy on the normalized chain complex $\overline{F}_\bullet = F_\bullet / D_\bullet$.
3. Identify a basis of $\overline{F}_\bullet$ in degrees $0, 1$, 2, and 3, and the differentials.
4. Let M be a $\mathbb{Z}G$-module. Identify normalized cocycles and coboundaries in $\mathrm{Hom}_{\mathbb{Z}G}(\overline{F}_\bullet; M)$ in degrees 0, 1, and 2.

◊ **Exercice 3. Extension of scalars.** Let $\alpha\colon R \to S$ be a ring homomorphism. We will write M for an R-module and N for an S-module.

1. **Extension of scalars.** Show that $S \otimes_R M$ is an S-module and the map $M \to S \otimes_R M$ defined by $m \mapsto 1 \otimes m$ is an R-module map.
2. **Universal property.** Any R-module map $f\colon M \to N$ factors through the extension of scalars map from point 1.
3. Apply Point 2 to the identity to obtain a *canonical* S-module map $S \otimes_R N \to N$ and show its surjectivity.
4. **Co-extension of scalars.** Dualize briefly to obtain a canonical injective S-module map $N \to \mathrm{Hom}_R(S, N)$.
5. Identify the extension and co-extension of scalars associated to the inclusion $\alpha\colon \mathbb{Z}H \to \mathbb{Z}G$.

◊ indicates the weekly assignments.